

Spherically symmetric brane in a bulk of $f(R)$ and Gauss-Bonnet Gravity

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Abstract

Starting from the effective gravitational field equations on a brane embedded in a bulk, described by Einstein-Hilbert plus the Gauss-Bonnet correction term, we have derived static spherically symmetric vacuum solutions. As expected, the solutions has the corresponding general relativity part and then additional corrections originating from the Gauss-Bonnet term. It turns out that spherically symmetric solution obtained from perturbative method for the same brane bulk configuration matches exactly with the solution derived from effective field equations. The same exercise has been carried forward for a bulk with $f(R)$ plus the Gauss-Bonnet corrections. In this situation as well the vacuum solutions derived from effective field equations and perturbative method matches *exactly*. Moreover it is shown that such higher curvature corrections modifies the character of vacuum solution drastically from their Einstein counterpart. For example, a black hole solution in pure general relativity can turn into a naked singularity or a new black hole solution due to the presence of the Gauss-Bonnet term in the Einstein-Hilbert action. Similar scenario is also observed when Gauss-Bonnet term is added to the $f(R)$ action in the bulk. Implications of these results are discussed.

1 Introduction

Higher dimensional spacetime is a natural working stage for string theory [1–3]. One of the string inspired models implementing this higher dimensional spacetime manifold for addressing a possible resolution of the long standing hierarchy problem is the brane world model [4–6]. In addition to resolving the gauge hierarchy problem (which essentially originates from the huge energy gap between the weak scale and the Planck scale), the brane world model can also address the issue of cosmological constant [7, 8]. Initially, the extra dimensions were assumed to be flat and compact,

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such that the volume of the extra dimensions suppresses the Planck scale in the bulk (the full higher dimensional spacetime) to TeV scale on the brane (which is the 4-dimensional hypersurface we live in). The only problem with this approach is that it does not include gravity, which being a universal (all particles feels it) and long range interaction must propagate on the bulk (for another novel justification see [9]). This problem was remedied immediately after, in [10] by invoking gravity in the bulk and thereby warping the extra dimensions such that the warp factor reduces the Planck scale to TeV scale on the visible brane. However the Randall Sundrum setup [10] uses two brane system with a finite separation between them. The origin of this finite separation (known as radion field) between the two branes was explained in [11] by introducing a bulk scalar field and then minimizing the potential resulting in stabilized value of the radion field. This stabilization of the radion field can be done in a dynamic situation as well and is very intimately related to the spacetime inflation at the early stage of the universe [12]. Since then there have been numerous investigations in the context of brane world scenario, e.g., in cosmology [12–21], in particle phenomenology [22–25], black hole formation and characterization [26–28]. In the Randall-Sundrum model a single warped extra dimension was used. Generalization of warped brane world models to higher dimensions, their phenomenology, cosmology and black hole solutions have been studied in [29–35].

As we have mentioned earlier that if gravity is considered in the bulk, it will induce its effect on the brane as well. Due to the existence of extra dimensions the effective gravitational field equations on the brane gets modified. When gravity in the bulk is described by general relativity the effective gravitational field equations induced on the brane was first derived in [36]. It turns out that there exist non-local bulk effects on the brane originating from the electric part of bulk Weyl tensor, which modifies the field equations for gravity. This in turn modifies the solutions to the field equations as well. The first such solution was derived in [37] and was subsequently generalized in [38]. In [35] the effective field equation for general relativity was derived for an m -brane embedded in a n -dimensional bulk, where m, n are arbitrary and thereby generalizing all the previous results in the context of general relativity. There we have presented both the spherically symmetric and cosmological solutions to the effective field equations. It may be noted that all the solutions were derived in the context of general relativity only. However there is a general belief that general relativity is only a low energy theory and thus it should inherit higher order corrections. Two most natural candidates in this direction are, Lanczos-Lovelock gravity and $f(\mathcal{R})$ gravity theory. Lanczos-Lovelock gravity is unique in the sense that even though it has higher order corrections the field equations are only second order in the dynamical variables [39–41]. Moreover from the thermodynamic perspectives as well the Lanczos-Lovelock gravity is preferred [42–46]. The first correction due to higher order Lanczos-Lovelock theories of gravity to the Einstein-Hilbert Lagrangian corresponds to the Gauss-Bonnet term. Thus it is natural to ask the behavior of effective field equations in presence of the Gauss-Bonnet term in the bulk, which has been studied in [47] and covariant effective gravitational equations have been obtained. However due to complicated nature of the field equations, only cosmological solutions to these field equations have been studied. Apart from this method for obtaining effective field equations by projecting bulk equation on the brane, there is another tool that can also be used to obtain field equations for gravity. This is a perturbative method with ratio of brane to bulk curvature as the perturbation parameter. In this method we solve for bulk equation in an iterative manner, i.e., we solve for n th order in the perturbation theory, plug that in the original equation and obtain solution for $(n+1)$ th order. The low energy effective gravitational field equations for Gauss-Bonnet gravity in the bulk were obtained in [48–52]. In this work we will use both the effective theory formalism and the low energy effective action in order to obtain static, spherically symmetric vacuum solutions with Gauss-Bonnet term

in the bulk. We will also discuss validity of these approaches as well as their differences.

On the other hand the $f(\mathcal{R})$ theories of gravity are important from another perspective. The cosmic acceleration at late times to inflationary scenario at early times are very well described by invoking a $f(\mathcal{R})$ term in the gravitational action. The $f(\mathcal{R})$ theory of gravity also passes through all the local tests (e.g., the solar system tests) for gravity and can be a strong candidate for gravity theory at high energy [53–57]. Also the $f(\mathcal{R})$ gravity models has interesting astrophysical applications, e.g., in the context of accretion [58] and neutrino spin and flavour oscillation [59, 60]. The effective equation formalism for $f(\mathcal{R})$ gravity has been discussed and spherically symmetric solutions were obtained in [61]. The effective equation formalism was subsequently generalized in [35] for arbitrary number of extra dimensions as well and also spherically symmetric and cosmological solutions to the effective field equations were obtained. In this work we will discuss the situation where both $f(R)$ and Gauss-Bonnet gravity propagates in the bulk, thus obtaining spherically symmetric solutions to the field equations through both the effective equation formalism and low energy effective action formalism.

The paper is organized as follows: We start with a brief introduction to the effective fields equations in Gauss-Bonnet and $f(\mathcal{R})$ gravity models and various physical interpretations thereof in Section 2. Then in Section 3 we have obtained spherically symmetric solutions to the effective field equations, derived for Einstein-Hilbert plus Gauss-Bonnet action.

2 Effective Gravitational Field Equations: A Brief Review

Our main aim in this work is to solve the effective field equations and obtain spherically symmetric solution. However before going into solving the field equations, we provide a brief introduction to the basic formalism. For this we will mainly follow [35, 47, 48, 61]. The full spacetime manifold is taken to be 5-dimensional, with a single 4-dimensional brane embedded within it. The five dimensional spacetime is characterized by the combination (\mathcal{M}, g_{ab}) and the action is described by,

$$\mathcal{A}_{\text{bulk}} = \int_{\mathcal{M}} d^5x \sqrt{-g} \left[\frac{1}{2\kappa_5^2} (\mathcal{R} + \alpha \mathcal{L}_{\text{GB}}) + \mathcal{L}_{\text{m}} \right] \quad (1)$$

where κ_5^2 represents the 5-dimensional gravitational constant, \mathcal{L}_{m} represents the matter Lagrangian and \mathcal{L}_{GB} represents the Gauss-Bonnet correction term and has the expression,

$$\mathcal{L}_{\text{GB}} = \mathcal{R}^2 - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}_{abcd}\mathcal{R}^{abcd} \quad (2)$$

In the above expressions \mathcal{R} is the Ricci scalar, \mathcal{R}_{ab} is the Ricci tensor, \mathcal{R}_{abcd} is the Riemann tensor while \mathcal{L}_{m} stands for the matter Lagrangian. α appearing in the above expressions is the Gauss-Bonnet coupling constant. Now we can consider the four-dimensional brane world to be a hypersurface Σ in the five-dimensional bulk spacetime with induced metric $h_{\mu\nu} = e_{\mu}^a e_{\nu}^b (g_{ab} - n_a n_b)$. Here $e_{\mu}^a = (\partial x^a / \partial y^{\mu})$, where x^a are the bulk coordinates and y^{μ} are defined on the brane. The vector n_a appearing in the definition of the induced metric is the unit normal to the hypersurface Σ . Also there would be matter present on the brane yielding a boundary action as well. When the brane plus bulk action is varied the gravitational field equations on the bulk turns out to be [39],

$$\mathcal{G}_{ab} + \alpha \mathcal{H}_{ab} = \kappa_5^2 [\mathcal{T}_{ab} + \tau_{ab} \delta(\Sigma)] \quad (3)$$

where we have used the following definitions,

$$\mathcal{G}_{ab} = \mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R} \quad (4)$$

$$\mathcal{H}_{ab} = 2 \left(\mathcal{R}\mathcal{R}_{ab} - 2\mathcal{R}_{ac}\mathcal{R}_b^c - 2\mathcal{R}^{cd}\mathcal{R}_{acbd} + \mathcal{R}_a^{cde}\mathcal{R}_{bcde} \right) - \frac{1}{2}g_{ab}\mathcal{L}_{\text{GB}} \quad (5)$$

and \mathcal{T}_{ab} is the matter energy-momentum tensor originating from the bulk matter action \mathcal{L}_m with τ_{ab} being brane energy momentum tensor, thanks to the $\delta(\Sigma)$ term.

Now the standard procedure to obtain the effective field equations on the brane is to use the Gauss-Codazzi relation [39] connecting bulk curvature tensors to the brane curvature tensors. This procedure in turn brings in the extrinsic curvature associated with the hypersurface Σ . Proceeding along these lines and writing the effective equations as Einstein equations plus correction terms we obtain [47],

$$\begin{aligned} G_{\mu\nu} + E_{\mu\nu} - K K_{\mu\nu} + K_{\mu\rho}K_\nu^\rho + \frac{1}{2} \left(K^2 - K_{\alpha\beta}K^{\alpha\beta} \right) h_{\mu\nu} + \alpha \left(H_{\mu\nu}^{(1)} + H_{\mu\nu}^{(2)} + H_{\mu\nu}^{(3)} \right) \\ = \frac{2\kappa_5^2}{3} \left[\left\{ \mathcal{T}_{ab}e_\mu^a e_\nu^b + \left(\mathcal{T}_{ab}n^a n^b - \frac{1}{4}\mathcal{T} \right) h_{\mu\nu} \right\} + \frac{\alpha}{3 + \alpha M} \left(M_{\mu\nu} - \frac{1}{4}M h_{\mu\nu} \right) \mathcal{T}_{ab}h^{ab} \right] \end{aligned} \quad (6)$$

In the above expression, $E_{\mu\nu}$ stands for the electric part of the bulk Weyl tensor projected on the brane. The three more tensors $H_{\mu\nu}^{(1)}$, $H_{\mu\nu}^{(2)}$ and $H_{\mu\nu}^{(3)}$ present in Eq. (6) stands for the various quadratic combinations of the curvature tensor, Weyl tensor and derivatives of extrinsic curvature. Since the expressions involved for these tensors are quite long, we have postponed their expressions till Appendix A (for detailed expressions see Eq. (38), Eq. (39) and Eq. (40) in Appendix A.1). Note that if we substitute $\alpha = 0$, then we would retrieve the effective equations for Einstein-Hilbert action.

The above effective field equations on a brane were derived on the premise when the bulk spacetime is endowed with Einstein-Hilbert Lagrangian plus a Gauss-Bonnet correction term. Given the current significance of the $f(\mathcal{R})$ gravity model another important situation arises if the first order term in the action is not the Einstein-Hilbert Lagrangian but a $f(\mathcal{R})$ term, where f is an arbitrary function of \mathcal{R} , the Ricci scalar of the bulk spacetime. Thus the action presented in Eq. (1) modifies to,

$$\mathcal{A}_{\text{bulk},f} = \int_{\mathcal{M}} d^5x \sqrt{-g} \left[\frac{1}{2\kappa_5^2} \{ f(\mathcal{R}) + \alpha \mathcal{L}_{\text{GB}} \} + \mathcal{L}_m \right] \quad (7)$$

with a similar contribution coming from the brane energy momentum tensor and brane tension. Now we can follow the same procedure, use the Gauss-Codazzi equations, relate brane curvature tensors to the bulk one and finally project the bulk gravitational field equations, using induced metric on Σ . After all these steps the effective gravitational field equations finally turn out to be (see [47] and [61])

$$\begin{aligned} G_{\mu\nu} + E_{\mu\nu} - F(\mathcal{R})h_{\mu\nu} - K K_{\mu\nu} + K_{\mu\rho}K_\nu^\rho + \frac{1}{2} \left(K^2 - K_{\alpha\beta}K^{\alpha\beta} \right) h_{\mu\nu} + \alpha \left(\hat{H}_{\mu\nu}^{(1)} + \hat{H}_{\mu\nu}^{(2)} + \hat{H}_{\mu\nu}^{(3)} \right) \\ = \frac{2\kappa_5^2}{3} \left[\left\{ \mathcal{T}_{ab}e_\mu^a e_\nu^b + \left(\mathcal{T}_{ab}n^a n^b - \frac{1}{4}\mathcal{T} \right) h_{\mu\nu} \right\} + \frac{\alpha}{3 + \alpha M} \left(M_{\mu\nu} - \frac{1}{4}M h_{\mu\nu} \right) \mathcal{T}_{ab}h^{ab} \right] \end{aligned} \quad (8)$$

where again $E_{\alpha\beta}$ stands for the electric part of bulk Weyl tensor projected on the brane. The tensors $H_{\mu\nu}^{(1)}$, $H_{\mu\nu}^{(2)}$ and $H_{\mu\nu}^{(3)}$ as in the case of Einstein-Hilbert plus Gauss-Bonnet corrections, contain various quadratic combinations of the curvature, Weyl and derivatives of extrinsic curvature tensor (for their detailed expressions see [Appendix A](#)). Also the term $F(\mathcal{R})$ appearing on the left hand side of [Eq. \(8\)](#) is connected to the Lagrangian term $f(\mathcal{R})$ through the following relation,

$$F(\mathcal{R}) = \left[\frac{1}{4} \frac{f(\mathcal{R})}{f'(\mathcal{R})} - \frac{1}{4} \mathcal{R} - \frac{2}{3} \frac{\square f'(\mathcal{R})}{f'(\mathcal{R})} + \frac{2}{3} \frac{\nabla_a \nabla_b f'(\mathcal{R})}{f'(\mathcal{R})} n^a n^b \right]_{\Sigma} \quad (9)$$

Here Σ in the subscript of [Eq. \(9\)](#) signifies that the quantity is evaluated on the hypersurface Σ , i.e., on the brane. As argued in [\[61\]](#) this term can be taken to be constant on the hypersurface Σ when the bulk Ricci scalar \mathcal{R} does not depend on x^μ , the brane coordinates. Note that this is precisely true for vacuum solutions on the brane since then the brane Ricci scalar identically vanishes. Note that for $f(\mathcal{R}) = \mathcal{R}$, $F(\mathcal{R})$ vanishes identically and we get back [Eq. \(6\)](#). In this as well, substitution of $\alpha = 0$ in [Eq. \(8\)](#) results in the effective equations for $f(\mathcal{R})$ gravity as presented in [\[61\]](#).

So far we have been discussing the effective equations approach. However as we have mentioned there is another useful approach to this problem, which is the low energy covariant curvature approach, in which the bulk equations are expanded perturbatively in the ratio of brane to bulk curvature. In this method, one starts with the bulk gravitational field equations and expand both the extrinsic curvature, curvature and Weyl tensor in a power series in the brane to bulk curvature ratio. Then writing the bulk equations for zeroth order, a zeroth order solution is obtained. This subsequently is inserted in the first order equation and the first order correction to the zeroth order solution is obtained, which is incorporated in the second order equations and so on. This method was first developed in [\[49–51\]](#) in the context of Einstein-Hilbert action. Later in [\[48\]](#) this formalism was extended for Gauss-Bonnet gravity as well. In this work we will confine ourselves by considering the effective field equations up to second order in the brane-bulk curvature ratio. The low energy effective equation for action presented in [Eq. \(1\)](#) turns out to be [\[48\]](#)

$$G_\mu^\nu = \frac{\kappa_5^2}{\ell(1+\beta)} T_\mu^\nu + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\bar{\chi}_\mu^\nu + \frac{(1-3\beta)\ell^2}{1+\beta} \mathcal{P}_\mu^\nu + \frac{\beta\ell^2}{1+\beta} C_{\mu\alpha}{}^{\nu\beta} R_\beta^\alpha - \frac{\beta\ell^2}{3} \left[\mathcal{W}_\mu^\nu - \frac{7}{16} \delta_\mu^\nu \mathcal{W} \right]. \quad (10)$$

where $\beta = (4\alpha/\ell^2)$ is a dimensionless quantity, with α being the Gauss-Bonnet coupling constant and ℓ being the bulk curvature radius. The tensor $T_{\mu\nu}$ on the right hand side of low energy effective field equations is the energy-momentum tensor on the brane, $R_{\alpha\beta}$ is the Ricci tensor on the brane

and $C_{\alpha\beta\mu\nu}$ is the four-dimensional Weyl tensor. The other tensors has the following expressions,

$$\mathcal{W}_\nu^\mu \equiv C_{\mu\alpha}{}^{\beta\sigma} C_{\beta\sigma}{}^{\nu\alpha}, \quad \mathcal{W} = C_{\mu\alpha}{}^{\beta\sigma} C_{\beta\sigma}{}^{\mu\alpha} \quad (11a)$$

$$\mathcal{P}_\mu^\nu \equiv \frac{1}{6} R R_\mu^\nu - \frac{1}{4} R_\mu^\alpha R_\alpha^\nu + \frac{1}{8} \delta_\mu^\nu R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{16} \delta_\mu^\nu R^2 \quad (11b)$$

$$\begin{aligned} \mathcal{S}_\mu^\nu &\equiv R_\mu^\alpha R_\alpha^\nu - \frac{1}{4} \delta_\mu^\nu R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} R \left(R_\mu^\nu - \frac{1}{4} \delta_\mu^\nu R \right) - \frac{1}{2} (D^\alpha D_\mu R_\alpha^\nu + D_\alpha D^\nu R_\mu^\alpha) \\ &\quad + \frac{1}{3} D_\mu D^\nu R + \frac{1}{2} D^2 R_\mu^\nu - \frac{1}{12} \delta_\mu^\nu D^2 R \end{aligned} \quad (11c)$$

$${}^{(2)}\bar{\chi}_\mu^\nu \equiv {}^{(2)}\chi_\mu^\nu + \frac{\ell^3}{4} \mathcal{S}_\mu^\nu + \frac{\beta \ell^3}{6(1-\beta)} \left(\mathcal{W}_\mu^\nu - \frac{1}{4} \delta_\mu^\nu \mathcal{W} \right) + \frac{\ell^3}{8} \left[R_\mu^\alpha R_\alpha^\nu - \frac{1}{3} R R_\mu^\nu - \frac{1}{4} \delta_\mu^\nu \left(R_\beta^\alpha R_\alpha^\beta - \frac{1}{3} R^2 \right) \right] \quad (11d)$$

Among these \mathcal{W}_ν^μ is quadratic in the Weyl tensor, ${}^{(2)}\bar{\chi}_{\mu\nu}$ depends on the quadratic terms of brane curvature and Weyl tensors, as well as on tensors originating from integration of bulk equations over the extra dimension, i.e., ${}^{(2)}\chi_\mu^\nu$ in Eq. (11d). The detailed expressions and derivations can be found in [48].

Finally the above result, being derived for the Einstein-Hilbert action plus the Gauss-Bonnet correction term, can be generalized in a straightforward manner to $f(\mathcal{R})$ plus the Gauss-Bonnet correction term. As we have seen earlier, this amounts to addition of a term depending on the $f(\mathcal{R})$ term to the left hand side of effective field equation. In this case the low energy effective field equations in presence of $f(\mathcal{R})$ gravity will take the following form,

$$\begin{aligned} G_\mu^\nu - F(\mathcal{R}) \delta_\mu^\nu &= \frac{\kappa_5^2}{\ell(1+\beta)} T_\mu^\nu + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\bar{\chi}_\mu^\nu + \frac{(1-3\beta)\ell^2}{1+\beta} \mathcal{P}_\mu^\nu \\ &\quad + \frac{\beta\ell^2}{1+\beta} C_{\mu\alpha}{}^{\nu\beta} R_\beta^\alpha - \frac{\beta\ell^2}{3} \left[\mathcal{W}_\mu^\nu - \frac{7}{16} \delta_\mu^\nu \mathcal{W} \right]. \end{aligned} \quad (12)$$

where $F(\mathcal{R})$ is given by Eq. (9). Having discussed various effective equations in the context of different gravity theories briefly, we will now start our main discussion, i.e., to obtain vacuum spherically symmetric solutions to the above equations.

3 Static, Spherically Symmetric Vacuum Brane with Einstein-Gauss-Bonnet Gravity in the Bulk

In this section we will be concentrating on Einstein-Hilbert action with a Gauss-Bonnet correction term. As mentioned in the introduction itself, we are interested in vacuum solutions of the effective field equations, which for this particular situation is given in Eq. (6). Even though the effective equations in Eq. (6) looks horrible, the spacetime being vacuum and static, spherically symmetric helps a lot and takes the effective equations to a tractable form. Since the spacetime is taken to be vacuum the brane energy-momentum tensor identically vanishes and so does the extrinsic curvature, due to the Junction conditions. Due to static and spherical symmetry the line element can be written as (for some interesting consequences see [62]),

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2 \quad (13)$$

Even with these symmetries the effective equations cannot be solved. This is due to the reason that the left hand side of Eq. (6) contains curvature tensor squared expressions. Hence the differential equations for ν and λ would be non-linear and second order in the radial derivatives. To obtain analytic solution, on top of these symmetries we keep terms linear in Gauss-Bonnet coupling parameter α . This immediately suggests that all the quadratic terms are already multiplied by α and thus all the quadratic terms will be replaced by their general relativity expressions. Then for vacuum solutions in general relativity the Ricci scalar and the Ricci tensor both vanishes. Thus for vacuum, spherically symmetric, static spacetime the effective field equations linear in Gauss-Bonnet coupling constant takes the following form:

$$G_{\mu\nu} + E_{\mu\nu} + \alpha \left\{ \frac{4}{3} R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - \frac{7}{12} h_{\mu\nu} (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}) - 4 R_{\mu\rho\nu\sigma} E^{\rho\sigma} \right\} = 0 \quad (14)$$

Here $E_{\mu\nu}$ is obtained by projecting the electric part of bulk Weyl tensor on the brane, $R_{\mu\nu\alpha\beta}$ stands for four-dimensional curvature components evaluated for the background Einstein-Hilbert action and α is the Gauss-Bonnet coupling parameter. The metric has two unknown parameters namely, λ and ν . To solve them, we require two differential equations which are supplied by the temporal part and radial part of Eq. (14) respectively. Writing these two components of Eq. (14) explicitly we obtain (for a derivation see Appendix A)

$$\begin{aligned} G_t^t + E_t^t = & -\alpha \left[\frac{1}{3} (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + \frac{1}{3} (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + \frac{1}{3} (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right. \\ & - \frac{7}{3} (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 - \frac{7}{3} (g^{rr})^2 (g^{\phi\phi})^2 (R_{r\phi r\phi})^2 - \frac{7}{3} (g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 \\ & \left. - 4g^{tt} R_{trtr} E^{rr} - 4g^{tt} R_{t\theta t\theta} E^{\theta\theta} - 4g^{tt} R_{t\phi t\phi} E^{\phi\phi} \right] \end{aligned} \quad (15)$$

and

$$\begin{aligned} G_r^r + E_r^r = & -\alpha \left[\frac{1}{3} (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + \frac{1}{3} (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + \frac{1}{3} (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right. \\ & - \frac{7}{3} (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 - \frac{7}{3} (g^{rr})^2 (g^{\phi\phi})^2 (R_{r\phi r\phi})^2 - \frac{7}{3} (g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 \\ & \left. - 4g^{tt} R_{trtr} E^{rr} - 4g^{tt} R_{t\theta t\theta} E^{\theta\theta} - 4g^{tt} R_{t\phi t\phi} E^{\phi\phi} \right] \end{aligned} \quad (16)$$

In the above expression the $G_{\mu\nu}^{\mu}$ on the left hand side is for the full Einstein-Hilbert plus Gauss-Bonnet gravity, while the terms on the right hand side being already linear in α are evaluated for background general relativity. Before going to evaluate the right hand side using general relativity solution let us spend a few moments explaining the electric tensor $E_{\mu\nu}$.

The electric part of the Weyl tensor $E_{\mu\nu} = e_{\mu}^a n^b e_{\nu}^c n^d C_{abcd}$ originates from the bulk Weyl tensor C_{abcd} . Thus it can inherit nonlocal effects from free bulk gravitational field. The tensor $E_{\mu\nu}$ is also expressible in terms of vectors in the four-dimensional spacetime. For that purpose we need the four velocity of a static observer and vector along radial direction, since we are interested in spherically symmetric spacetime. This decomposition has been elaborated for the general case in [63] and the projected Weyl tensor can be expanded as,

$$E_{\mu\nu} = - \left(\frac{k_5}{k_4} \right)^4 \left[U(r) \left(u_{\mu} u_{\nu} + \frac{1}{3} \xi_{\mu\nu} \right) + P_{\mu\nu} + 2Q_{(\mu} u_{\nu)} \right] \quad (17)$$

with $k_4^2 = 8\pi G_N$ being the four-dimensional gravitational constant. Among the other tensors appearing in Eq. (17) we have $\xi_{\mu\nu} = h_{\mu\nu} + u_\mu u_\nu$ as the induced metric on a $t = \text{constant}$ hypersurface. The other terms present in Eq. (17) are respectively, the “Dark Radiation” term, $U = -(k_4/k_5)^4 E_{\mu\nu} u^\mu u^\nu$, which is a scalar, $Q_\mu = (k_4/k_5)^4 \xi_\mu^\alpha E_{\alpha\beta}$ is a spatial vector and $P_{\mu\nu} = -(k_4/k_5)^4 \{\xi_{(\mu}^\alpha \xi_{\nu)}^\beta - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta}\} E_{\alpha\beta}$ is a spatial, trace free, symmetric tensor. For static solutions, $Q_\mu = 0$ and the tensor $P_{\mu\nu} = P(r) (r_\mu r_\nu - \frac{1}{3} \xi_{\mu\nu})$, where r_μ stands for the unit radial vector and $P(r)$ is known as the “Dark Pressure”.

Let us now resume our main job, i.e., to find a static, spherically symmetric vacuum solution to the effective field equations Eq. (6). For that we need a background general relativity vacuum solution. This was obtained in [37] and the metric elements in the light of Eq. (13) takes the form,

$$e^\nu = e^{-\lambda} = 1 - \frac{2GM + Q_0}{r} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{1}{r^2} \equiv 1 - \frac{a}{r} - \frac{b}{r^2} \quad (18)$$

where Q_0 is an integration constant and is related to the total dark radiation within a spherical volume and λ_T stands for the brane tension. The two unknown functions $U(r)$ and $P(r)$ has the following functional behavior: $U(r) = -(P_0/2r^4)$ and $P(r) = (P_0/r^4)$, which brings P_0 in Eq. (18). The two constants a and b are determined from the general relativity solution and have respective values: $a = 2GM + Q_0$ and $b = (3P_0/8\pi G\lambda_T)$. We now compute the components of curvature tensor using the background general relativity solution discussed earlier and obtain an expression for the right hand side of effective field equations as presented in Eq. (15) and Eq. (16). This will lead to first order linear differential equations in ν and λ correct up to linear order in Gauss-Bonnet coupling constant. Thus performing this procedure (lengthy but straightforward; see Appendix A) we arrive at the following equation for the temporal component (for a detailed derivation see Eq. (64) in Appendix A.2),

$$\begin{aligned} G_t^t + 3\bar{\kappa}U(r) - \alpha \left(3\frac{a^2}{r^6} + \frac{20}{3}\frac{ab}{r^7} + \frac{10}{3}\frac{b^2}{r^8} \right) - 4\alpha \left[-\bar{\kappa} \left(\frac{a}{r^3} + \frac{3b}{r^4} \right) (U + 2P) \right. \\ \left. + \bar{\kappa} \frac{1}{2r} (U - P) \left(\frac{a}{r^2} + \frac{2b}{r^3} \right) + \bar{\kappa} \frac{1}{2r} (U - P) \left(\frac{a}{r^2} + \frac{2b}{r^3} \right) \right] = 0 \end{aligned} \quad (19)$$

where the constant $\bar{\kappa} = (1/3)(k_5/k_4)^4 = (1/4\pi G\lambda_T)$ [44] is related to the coefficient of electric part of Weyl tensor (see Eq. (17)). In the above expression we also have the following solutions for the dark radiation $U = -P_0/2r^4$ and dark pressure $P = P_0/r^4$. These results upon substitution in Eq. (19) leads to

$$G_t^t = e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = \frac{1}{r^2} \frac{d}{dr} (r e^{-\lambda}) - \frac{1}{r^2} \quad (20)$$

This immediately implies the following differential equation for $e^{-\lambda}$,

$$\frac{d}{dr} (r e^{-\lambda}) = 1 + \frac{3\bar{\kappa}P_0}{2r^2} - \frac{6\alpha\bar{\kappa}P_0}{r^2} \left(\frac{2a}{r^3} + \frac{5b}{r^4} \right) + \alpha \left(3\frac{a^2}{r^4} + \frac{20}{3}\frac{ab}{r^5} + \frac{10}{3}\frac{b^2}{r^6} \right) \quad (21)$$

The left hand side is a linear differential operator acting on $e^{-\lambda}$, the g_{rr} metric component and the right hand is solely a function of r , due to spherical symmetry. This can be readily integrated and

finally we obtain:

$$\begin{aligned}
e^{-\lambda} &= 1 - \frac{a}{r} - \frac{b}{r^2} - \alpha \left(\frac{a^2}{r^4} + \frac{5ab}{3r^5} + \frac{2b^2}{3r^6} - \frac{3\bar{\kappa}P_0a}{r^5} - \frac{6\bar{\kappa}P_0b}{r^6} \right) \\
&= \left\{ 1 - \frac{2GM + Q_0}{r} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{1}{r^2} \right\} \\
&\quad - \alpha \left\{ \frac{(2GM + Q_0)^2}{r^4} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{2GM + Q_0}{3r^5} - \left(\frac{3P_0}{8\pi G\lambda_T} \right)^2 \frac{10}{3r^6} \right\} \quad (22)
\end{aligned}$$

where in obtaining the last line we have incorporated the values of a , b and $\bar{\kappa}$ respectively, which are, $a = 2GM + Q_0$, $b = 3P_0/8\pi G\lambda_T$ and $\bar{\kappa} = (1/4\pi G\lambda_T)$.

We note that the introduction of additional Gauss-Bonnet correction term alters the nature of the black hole solution considerably. The most prominent change corresponds to change in the location of the black hole horizon. In the original spacetime the horizon is located at the radius where $(1 - (a/r) - (b/r^2))$ identically vanishes. However in the modified metric the above horizon is no longer the radius on which $f(r)$ vanishes. In this modified metric the solution of $f(r) = 0$ corresponds to a sixth degree algebraic equation, whose solution differs widely from the original location of the black hole horizon.

Along similar lines we can use the background general relativity metric elements to compute the right hand side of Eq. (16). Then it will again provide a linear differential equation in e^ν . Since due to spherical symmetry the right hand side of Eq. (16) depends only on the radial coordinate r , that can be readily integrated leading to solution for e^ν . The differential equation for e^ν originating from G_r^r turns out to be (for a detailed derivation see Eq. (65) of Appendix A.2),

$$\begin{aligned}
e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} &= \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{1}{r^4} + \alpha \left[\left\{ 3 \frac{(2GM + Q_0)^2}{r^6} + \frac{5(2GM + Q_0)P_0}{2\pi G\lambda_T} \frac{1}{r^7} + \frac{10}{3} \left(\frac{3P_0}{8\pi G\lambda_T} \right)^2 \frac{1}{r^8} \right\} \right. \\
&\quad \left. - \frac{3P_0}{2\pi G\lambda_T} \frac{1}{r^4} \left\{ 2 \frac{2GM + Q_0}{r^3} + 5 \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{1}{r^4} \right\} \right] \quad (23)
\end{aligned}$$

Then finally integrating this expression we readily obtain the solution for g_{tt} component as $e^\nu = e^{-\lambda}$, where $e^{-\lambda}$ is given by Eq. (22). In the background spacetime described by general relativity we had $g_{tt} = g_{rr}^{-1}$, then it turns out that even when Gauss-Bonnet correction term is added to the Einstein-Hilbert Lagrangian, to first order in Gauss-Bonnet coupling parameter, the vacuum solution still satisfies the condition $g_{tt} = g_{rr}^{-1}$.

Having obtained the vacuum spherically symmetric static solution for Einstein-Hilbert action with Gauss-Bonnet correction term to linear order in the Gauss-Bonnet coupling parameter, we will now take up the case of $f(\mathcal{R})$ gravity added to Gauss-Bonnet correction term and shall obtain the vacuum solution thereof.

4 Vacuum Solution in the brane with bulk $f(\mathcal{R})$ and Gauss-Bonnet Gravity

Let us now take up the case of $f(\mathcal{R})$ gravity and Gauss-Bonnet gravity present in the bulk, such that the bulk gravitational action takes the form presented in Eq. (7). In this case also our main

interest will be to study the vacuum solutions. We will also impose the condition that spacetime should be static and spherically symmetric. This would imply that the radial and time part of the metric elements should be independent of time coordinate and they can only depend on radial coordinate r . Thus the line element can be written as [Eq. \(13\)](#), with $g_{tt} = e^{\nu(r)}$, $g_{rr} = e^{\lambda(r)}$.

In this case also, these symmetries cannot help to analytically solve the problem, since the differential equations involved are second order and non-linear. We once again impose conditions, to take up to linear order in α , the Gauss-Bonnet coupling constant. In this case we need to solve for the background general relativity solution and then compute all the quadratic terms in the curvature tensor. This implies that all the quadratic terms with Ricci tensor and Ricci scalar vanishes. This leads to the following form for the effective field equations,

$$G_{\mu\nu} + E_{\mu\nu} - F(\mathcal{R})h_{\mu\nu} = -\alpha \left\{ \frac{4}{3} R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - \frac{7}{12} h_{\mu\nu} (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}) - 4 R_{\mu\rho\nu\sigma} E^{\rho\sigma} \right\} \quad (24)$$

where again $E_{\mu\nu}$ stands for the electric part of the bulk Weyl tensor, which can be decomposed into a dark radiation part and a dark pressure part as in [Eq. \(17\)](#). $R_{\alpha\beta\mu\nu}$ stands for Riemann tensor on the brane. Having obtained the effective field equation, we shall now proceed to determine the solution for the two unknown parameters, namely λ and ν . Along similar lines as in the previous section, we will work to linear orders in Gauss-Bonnet coupling parameter α . This immediately suggests that the right hand side expressions in the effective field equations are to be evaluated for background $f(\mathcal{R})$ gravity model. Such a solution was obtained in [\[61\]](#) and corresponds to the following line element,

$$ds^2 = 1 - \frac{2GM + Q_0}{r} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{1}{r^2} + \frac{F(\mathcal{R})}{3} r^2 \equiv 1 - \frac{a}{r} - \frac{b}{r^2} + cr^2 \quad (25)$$

Here Q_0 stands for an integration constant and is related to the total dark radiation within a spherical volume. λ_T denotes the brane tension. The origin of P_0 in [Eq. \(25\)](#) is from the dark radiation term i.e., from the bulk Weyl tensor. The three constants a , b and c are determined from the $f(\mathcal{R})$ solution to have the respective values: $a = 2GM + Q_0$, $b = (3P_0/8\pi G\lambda_T)$ and $c = F(\mathcal{R})/3$. Having obtained the metric it is straightforward to compute the components of curvature tensor and obtain an expression for the right hand side of the effective field equations as presented in [Eq. \(24\)](#). This will eventually lead to first order linear differential equations in ν and λ correct up to linear order in Gauss-Bonnet coupling constant. Performing this procedure (see [Appendix A](#)) we arrive at the following differential equation for $e^{-\lambda}$ (see [Eq. \(76\)](#) in [Appendix A.2](#) for a detailed derivation),

$$\begin{aligned} \frac{d}{dr}(re^{-\lambda}) = & 1 + \frac{3\bar{\kappa}P_0}{2r^2} + \alpha \left(3\frac{a^2}{r^4} + \frac{20}{3}\frac{ab}{r^5} + \frac{10}{3}\frac{b^2}{r^6} + \frac{20}{3}c^2r^2 + \frac{16}{3}\frac{bc}{r^2} \right) \\ & + F(\mathcal{R})r^2 - \frac{6\alpha\bar{\kappa}P_0}{r^2} \left(\frac{2a}{r^3} + \frac{5b}{r^4} + c \right) \end{aligned} \quad (26)$$

where we have introduced this quantity $\bar{\kappa} = (1/4\pi G\lambda_T)$. The above equation of $e^{-\lambda}$, being an ordinary, first order differential equation can be readily integrated and thus a solution for $e^{-\lambda}$ can

be obtained. This solution turns out to be,

$$\begin{aligned}
e^{-\lambda} &= 1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + F(\mathcal{R})\frac{r^2}{3} - \alpha \left(\frac{a^2}{r^4} - \frac{ab}{3r^5} - \frac{10b^2}{3r^6} - \frac{20}{9}c^2r^2 + \frac{4}{3}\frac{bc}{r^2} \right) \\
&= \left(1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + F(\mathcal{R})\frac{r^2}{3} \right) \\
&\quad - \alpha \left[\frac{(2GM + Q_0)^2}{r^4} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{2GM + Q_0}{3r^5} - \left(\frac{3P_0}{8\pi G\lambda_T} \right)^2 \frac{10}{3r^6} - \frac{20}{81}F(\mathcal{R})^2r^2 + \frac{4}{3}\frac{P_0F(\mathcal{R})}{8\pi G\lambda_T r^2} \right]
\end{aligned} \tag{27}$$

where in order to obtain the last line we have substituted for $a = 2GM + Q_0$, $b = (3P_0/8\pi G\lambda_T)$ and $c = F(\mathcal{R})/3$. Also note that for $e^{-\lambda}$ the first four terms constitute the standard $f(\mathcal{R})$ gravity solution [61], plus additional correction terms of order α , the Gauss-Bonnet coupling parameter, as the action is modified by the addition of a Gauss-Bonnet term.

We note that due to this addition of Gauss-Bonnet correction term to the original Einstein-Hilbert action the location of the black hole horizon changes drastically. A black hole solution in the original spacetime can be altered to a different location or even can change it to a naked singularity. This is evident, since the original fourth order algebraic equation gets transformed to a sixth order one and depending on the parameter space of the parameters this may not have a real solution, leading to existence of naked singularity.

Now we have to solve for e^ν in order to obtain the full spherically symmetric solution. For that we require the radial part of the effective equations. Like e^λ , for e^ν as well we have first order and ordinary differential equation. This differential equations turns out to have the following form (see Eq. (79) in Appendix A.3 for detailed discussion),

$$\begin{aligned}
e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} &= F(\mathcal{R}) + \frac{3P_0}{8\pi G\lambda_T} \frac{1}{r^4} + \alpha \left[\left\{ \frac{3(2GM + Q_0)^2}{r^6} + \frac{10}{3} \left(\frac{3P_0}{8\pi G\lambda_T} \right)^2 \frac{1}{r^8} \right. \right. \\
&\quad \left. \left. + \frac{20}{27}F(\mathcal{R})^2 + \frac{20}{3} \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{2GM + Q_0}{r^7} + \frac{10}{9} \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{F(\mathcal{R})}{r^4} \right\} \right. \\
&\quad \left. - \frac{3P_0}{2\pi G\lambda_T} \frac{1}{r^4} \left\{ \frac{2(2GM + Q_0)}{r^3} + \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{5}{r^4} + \frac{F(\mathcal{R})}{3} \right\} \right]
\end{aligned} \tag{28}$$

This equation can be integrated and we readily obtain the solution for g_{tt} component as $e^\nu = e^{-\lambda}$, where $e^{-\lambda}$ is given by Eq. (27). In the background spacetime described by $f(\mathcal{R})$ gravity we had $g_{tt} = g_{rr}^{-1}$. To our surprise, it turns out that even when Gauss-Bonnet correction term is added to the $f(\mathcal{R})$ Lagrangian, to first order in Gauss-Bonnet coupling parameter the vacuum solution still satisfies the condition $g_{tt} = g_{rr}^{-1}$.

So far we have obtained the vacuum spherically symmetric static solution for both the Einstein-Hilbert action and $f(\mathcal{R})$ action with Gauss-Bonnet correction term. Both these solutions are obtained to linear order in the Gauss-Bonnet coupling parameter. We will now take up the case of low energy effective equations and spherically symmetric solutions obtained from them.

5 Low Energy Effective Field Equations on the brane with Einstein-Gauss-Bonnet Gravity in the Bulk

In the previous sections we have derived the spherically symmetric solutions starting from the effective field equations. These effective field equations are obtained by projecting bulk equations on the brane, using appropriate projection tensors. In this section we will try to obtain static spherically symmetric solution using low energy perturbative field equations. These are obtained by expanding the bulk equations with bulk curvature to intrinsic brane curvature ratio as a perturbative parameter. In principle these two approaches are fundamentally *different* and thus can give rise to different solutions.

For Einstein-Hilbert action with Gauss-Bonnet correction term, the low energy effective field equations have been presented in Eq. (10). Starting from which we should be able to derive equations for both the metric elements e^ν and e^λ respectively. However the right hand side of the effective equation in absence of external matter fields and for spherically symmetric spacetime becomes linear in the Gauss-Bonnet coupling parameter. Since we are interested in first order corrections to the spherically symmetric solution due to addition of the Gauss-Bonnet term all the quadratic curvature components can be evaluated for the background general relativity spacetime. This again suggests the form of the background line element to be given by Eq. (18). This has the following form $e^\nu = e^{-\lambda} = 1 - (a/r) - (b/r^2)$, with $a = 2GM + Q_0$ and $b = (3P_0/8\pi G\lambda_T)$. Still, we have an ambiguity in the low energy theory in the quantity ${}^{(2)}\bar{\chi}_\nu^\mu$. This is fixed by noting that ${}^{(2)}E_\nu^\mu = -(2/\ell) {}^{(2)}\bar{chi}_\nu^\mu$. Thus the quantity appearing in the field equations can be related to the electric part of the Weyl tensor. Using which in the quadratic curvature components we finally obtain the differential equation for $e^{-\lambda}$ to be (see Eq. (86) in Appendix A.4 for derivation),

$$\begin{aligned} \frac{d}{dr} (re^{-\lambda}) &= 1 + \frac{3\bar{\kappa}P_0}{2r^2} + \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t(r) r^2 \\ &\quad - \frac{\beta\ell^2}{12} \left(\frac{3a^2}{2r^4} + \frac{8ab}{r^5} + \frac{11b^2}{r^6} \right) + \frac{7\beta\ell^2}{12} \left(\frac{3a^2}{2r^4} + \frac{4ab}{r^5} + \frac{3b^2}{r^6} \right) \end{aligned} \quad (29)$$

where $a = 2GM + Q_0$, $b = (3P_0/8\pi G\lambda_T)$ and ${}^{(2)}\chi_t^t$ is dependent on the radial coordinate only and originates from integration of bulk field equations. Note that all the terms except for the first two are dependent linearly on β and hence on the Gauss-Bonnet coupling parameter α . This equation being an ordinary first order differential equation in $e^{-\lambda}$ can be readily integrated leading to (see Eq. (87) in Appendix A.4 for derivation),

$$e^{-\lambda} = 1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + \frac{16\alpha}{\ell^3} \frac{1}{r} \int {}^{(2)}\chi_t^t(r) r^2 dr - \frac{\alpha}{3} \left(3\frac{a^2}{r^4} + \frac{5ab}{r^5} + \frac{2b^2}{r^6} \right) \quad (30)$$

where we have written β in terms of the Gauss-Bonnet coupling parameter α and have defined $\bar{\kappa} = (1/4\pi G\lambda_T)$. It is clear that for $\alpha = 0$, we retrieve the standard solution of general relativity. All the correction terms are linear in α and originates from the addition of Gauss-Bonnet term to the original Einstein-Hilbert action. However still this solution has some arbitrariness, in the sense that the object ${}^{(2)}\chi_t^t$ has not been determined. The most remarkable feature of this solution comes into light if we choose the bulk integration constant ${}^{(2)}\chi_t^t$ to be,

$${}^{(2)}\chi_t^t = \ell^3 \left(-\frac{ab}{2r^7} - \frac{5b^2}{4r^8} \right) \quad (31)$$

In which case the solution for $e^{-\lambda}$ can be obtained exactly by substituting for $a = 2GM + Q_0$, $b = (3P_0/8\pi G\lambda_T)$ and $\bar{\kappa} = (1/4\pi G\lambda_T)$, leading to,

$$e^{-\lambda} = \left\{ 1 - \frac{2GM + Q_0}{r} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{1}{r^2} \right\} - \alpha \left\{ \frac{(2GM + Q_0)^2}{r^4} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{2GM + Q_0}{3r^5} - \left(\frac{3P_0}{8\pi G\lambda_T} \right)^2 \frac{10}{3r^6} \right\} \quad (32)$$

which matches *exactly* with the solution obtained from the full effective equation. Not only for $e^{-\lambda}$, by solving for the radial equation from the low energy effective equation, we obtain $e^\nu = e^{-\lambda}$. Hence the considerations regarding black hole horizon applies to this case as well, viz., a black hole solution in original spacetime can be transformed into either a black hole or a naked singularity solution.

Hence the two completely different procedure yields *identical* solution. This we have seen in the cosmological context as well [12]. This again shows that the effective equations obtained via Gauss-Codazzi equation, i.e, by projecting the bulk equations on the brane is related at some deeper level to the low energy effective equation obtained via a perturbative expansion of the bulk equation. Moreover, this approach can be extended to all orders. As we go to higher and higher orders new integration constants corresponding to the integration of bulk equations would appear. At each level we can tune these integration constants in order to match with the effective equation formalism. Hence this equality is not only valid for a particular order in the perturbative expansion, it will continue to hold at *all* orders of the perturbative expansion. This in turn implies that the modification in each order can be absorbed by redefining the expression for arbitrary integration constants (i.e., does not depend on extra coordinates) appearing in each order in the perturbative expansion. In the next section we will discuss the case of $f(\mathcal{R})$ gravity theory.

6 Low Energy Effective Field Equations on the Vacuum Brane with $f(\mathcal{R})$ and Gauss-Bonnet Gravity on the Bulk

In the previous section we have discussed the effect of Gauss-Bonnet correction term on the low energy effective field equations on the brane and the corrections in the respective spherically symmetric vacuum solutions. The results were derived to linear order in the Gauss-Bonnet coupling parameter α and shows exact similarity with the solution derived from the Gauss-Codazzi equations. This result was derived for Einstein-Hilbert action with a Gauss-Bonnet correction term, in this section we will try to see whether the same result holds for $f(\mathcal{R})$ gravity with Gauss-Bonnet correction term as well.

The low energy effective field equations for $f(\mathcal{R})$ gravity with Gauss-Bonnet correction term is presented in Eq. (12). Starting from which we need differential equations for e^λ and e^ν , which when solved would lead to the metric. Alike the previous cases, in this section as well we will keep terms linear in the Gauss-Bonnet coupling strength. This can be achieved by writing the quadratic curvature invariants in terms of the background $f(\mathcal{R})$ gravity spacetime for which the metric is given in Eq. (25). The metric can be written in the following form, $e^\nu = e^{-\lambda} = 1 - (a/r) - (b/r^2) + cr^2$, where $a = 2GM + Q_0$, $b = (3P_0/8\pi G\lambda_T)$ and $c = F(\mathcal{R})/3$. Here as well the ambiguity in ${}^{(2)}\bar{\chi}_\nu^\mu$ can be removed by setting ${}^{(2)}E_\nu^\mu = -(2/\ell) {}^{(2)}\bar{\chi}_\nu^\mu$. After doing all this, finally the differential equation

for $e^{-\lambda}$ obtained from low energy effective action turns out to be (see Eq. (93) of Appendix A.5 for a detailed discussion),

$$\begin{aligned} \frac{d}{dr} (re^{-\lambda}) &= 1 + \frac{3\bar{\kappa}P_0}{2r^2} + F(\mathcal{R})r^2 + \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t(r) r^2 - \frac{\beta\ell^2}{12} \left[\frac{3a^2}{2r^4} + \frac{11b^2}{r^6} + \frac{8ab}{r^5} + 3c^2r^2 - \frac{2bc}{r^2} \right] \\ &\quad + \frac{7\beta\ell^2}{12} \left[\frac{3a^2}{2r^4} + \frac{3b^2}{r^6} + \frac{4ab}{r^5} + 3c^2r^2 - \frac{2bc}{r^2} \right] \end{aligned} \quad (33)$$

where $\bar{\kappa} = (1/4\pi G\lambda_T)$. This differential equation for $e^{-\lambda}$ being an ordinary first order equation can be easily integrated. Under integration the metric element $e^{-\lambda}$ leads to the following expression

$$\begin{aligned} e^{-\lambda} &= 1 - \frac{2GM + Q_0}{r} - \frac{3P_0}{8\pi G\lambda_T} \frac{1}{r^2} + \frac{F(\mathcal{R}) - \Lambda_4}{3} r^2 + \frac{16\alpha}{\ell^3} \frac{1}{r} \int {}^{(2)}\chi_t^t(r) r^2 dr \\ &\quad - \frac{\alpha}{3} \left[\frac{3a^2}{r^4} + \frac{2b^2}{r^6} + \frac{5ab}{r^5} - 6c^2r^2 - \frac{12bc}{r^2} \right] \end{aligned} \quad (34)$$

In this connection we should mention that the differential equation for e^ν can be obtained from the radial part of the low energy effective gravitational equations. Like previous situations, the radial equation yields the same differential equation as of e^λ , leading to $e^\nu = e^{-\lambda}$. In this solution also we have an unknown function ${}^{(2)}\chi_t^t(r)$ for which the following choice,

$${}^{(2)}\chi_t^t = \ell^3 \left(-\frac{ab}{r^7} - \frac{5b^2}{4r^8} + \frac{bc}{3r^4} \right) \quad (35)$$

would lead the solution for $e^{-\lambda}$ to the following form,

$$\begin{aligned} e^{-\lambda} &= \left(1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + F(\mathcal{R})\frac{r^2}{3} \right) \\ &\quad - \alpha \left[\frac{(2GM + Q_0)^2}{r^4} - \left(\frac{3P_0}{8\pi G\lambda_T} \right) \frac{2GM + Q_0}{3r^5} - \left(\frac{3P_0}{8\pi G\lambda_T} \right)^2 \frac{10}{3r^6} - \frac{20}{81} F(\mathcal{R})^2 r^2 + \frac{4}{3} \frac{P_0 F(\mathcal{R})}{8\pi G\lambda_T r^2} \right] \end{aligned} \quad (36)$$

where we have substituted for the constants a, b and c with their respective values as, $a = 2GM + Q_0$, $b = (3P_0/8\pi G\lambda_T)$ and $c = F(\mathcal{R})/3$. Note that this is exactly the one we have obtained in Eq. (27). Hence even for $f(\mathcal{R})$ gravity with a Gauss-Bonnet correction term, static spherically symmetric solutions obtained from — (1) projection of bulk equations and thus obtaining an effective equation, and (2) from perturbative expansion of bulk equations in brane to bulk curvature ratio — match identically. This again signals some inherent deep connection between these two formulations of obtaining effective field equations on a lower dimensional hypersurface. Moreover, the exact nature was proved using second order perturbation theory. If we had kept higher order terms, there would more arbitrary functions like ${}^{(2)}\chi_t^t(r)$, which can be adjusted in order to match with solutions obtained from projecting bulk field equations.

7 Discussion

In the literature, there exist two possible ways to derive effective gravitational field equations on a brane starting from the higher dimensional bulk equations. One of them is achieved by projecting

the bulk geometrical quantities on the brane, which is accomplished through use of Gauss-Codazzi equations and their various contractions. The second one is obtained through a perturbative expansion of the bulk equations with brane to bulk curvature ratio as the perturbation parameter. These two approaches are based on completely different setup.

In this work we embark to show that this two approaches can lead to identical results in some simplified scenarios. Since addition of a higher curvature term to the Einstein-Hilbert action can lead to non-trivial results, we start with Einstein-Hilbert action with a Gauss-Bonnet correction term. From the projected effective equations we calculate corrections to the background static spherically symmetric metric to linear order in the Gauss-Bonnet coupling parameter. These additional terms can change the original black hole solution to either a black hole solution or a naked singularity. Hence the background geometry can be significantly distorted by addition of a higher curvature term to the Einstein-Hilbert action. More importantly, when similar exercise is performed using the low energy effective field equations in this perturbative approach, that also lead to *identical* solutions. To our surprise this feature continues to hold even for bulk $f(\mathcal{R})$ gravity with a Gauss-Bonnet correction term. This shows that these two apparently independent methods of getting the effective field equations yield identical solutions in the context of both Einstein-Hilbert and $f(\mathcal{R})$ action with a Gauss-Bonnet correction term.

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A Appendix: Detailed Derivations

In this appendix, we summarize all the detailed derivations of respective expressions presented in the main text. We hope this will be helpful to clarify all the algebraic steps necessary to arrive at definite results in the main body of this paper.

A.1 Basic Ingredients

The effective field equations obtained through projection of bulk gravitational field equations and Gauss-Codazzi equation takes the following form:

$$\begin{aligned} G_{\mu\nu} + E_{\mu\nu} - K K_{\mu\nu} + K_{\mu\rho} K_{\nu}^{\rho} + \frac{1}{2} (K^2 - K_{\alpha\beta} K^{\alpha\beta}) h_{\mu\nu} + \alpha (H_{\mu\nu}^{(1)} + H_{\mu\nu}^{(2)} + H_{\mu\nu}^{(3)}) \\ = \frac{2\kappa_5^2}{3} \left[\left\{ \mathcal{T}_{ab} e_{\mu}^a e_{\nu}^b + \left(\mathcal{T}_{ab} n^a n^b - \frac{1}{4} \mathcal{T} \right) h_{\mu\nu} \right\} + \frac{\alpha}{3 + \alpha M} \left(M_{\mu\nu} - \frac{1}{4} M h_{\mu\nu} \right) \mathcal{T}_{ab} h^{ab} \right] \end{aligned} \quad (37)$$

where most of the tensors have been discussed in detail in the main text with explicit expressions given. However the three tensors $H_{\mu\nu}^{(1)}$, $H_{\mu\nu}^{(2)}$ and $H_{\mu\nu}^{(3)}$ have been discussed but detailed expressions

were not provided in the main text, the same appears below,

$$\begin{aligned}
H_{\mu\nu}^{(1)} &= \frac{4}{3} (M_{\mu\alpha\beta\gamma} M_{\nu}^{\alpha\beta\gamma} - 3M^{\rho\sigma} M_{\mu\rho\nu\sigma} + 2M M_{\mu\nu} - 4M_{\mu\rho} M_{\nu}^{\rho}) \\
&\quad - \frac{1}{12} h_{\mu\nu} (11M^2 - 40M_{\alpha\beta} M^{\alpha\beta} + 7M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta}) \\
&\quad - \frac{\alpha}{3(3+\alpha M)} \left(M_{\mu\nu} - \frac{1}{4} M h_{\mu\nu} \right) (M^2 - 8M_{\alpha\beta} M^{\alpha\beta} + M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta}), \tag{38}
\end{aligned}$$

$$\begin{aligned}
H_{\mu\nu}^{(2)} &= -4 (M_{\mu\rho} E_{\nu}^{\rho} + M_{\nu\rho} E_{\mu}^{\rho} + M_{\mu\rho\nu\sigma} E^{\rho\sigma}) + 3h_{\mu\nu} M_{\rho\sigma} E^{\rho\sigma} + 2M E_{\mu\nu} \\
&\quad + \frac{4\alpha}{3+\alpha M} \left(M_{\mu\nu} - \frac{1}{4} M h_{\mu\nu} \right) M_{\rho\sigma} E^{\rho\sigma}, \tag{39}
\end{aligned}$$

$$\begin{aligned}
H_{\mu\nu}^{(3)} &= \frac{8}{3} \left[-N_{\mu} N_{\nu} + N^{\rho} (N_{\rho\mu\nu} + N_{\rho\nu\mu}) + \frac{1}{2} N_{\rho\sigma\mu} N^{\rho\sigma}_{\nu} + N_{\mu\rho\sigma} N_{\nu}^{\rho\sigma} \right] \\
&\quad + \left[2h_{\mu\nu} + \frac{8\alpha}{3(3+\alpha M)} \left(M_{\mu\nu} - \frac{1}{4} M h_{\mu\nu} \right) \right] \left(N_{\alpha} N^{\alpha} - \frac{1}{2} N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right). \tag{40}
\end{aligned}$$

Next we will discuss the field equations for spherically symmetric spacetime derived from the effective equation presented in [Eq. \(37\)](#).

A.2 Effective Field Equations in General Relativity with Gauss-Bonnet correction: Detailed Analysis

We are interested in static spherically symmetric spacetime for which the line element takes the form,

$$ds^2 = e^{\nu} dt^2 + e^{-\lambda} dr^2 + r^2 d\Omega^2 \tag{41}$$

The corresponding connections are:

$$\Gamma_{rt}^t = \frac{\nu'}{2}; \quad \Gamma_{rr}^t = \frac{\lambda}{2} e^{\lambda-\nu} \tag{42}$$

$$\Gamma_{rr}^r = \frac{\lambda'}{2}; \quad \Gamma_{\theta\theta}^r = -r e^{-\lambda}; \quad \Gamma_{tt}^r = \frac{\nu'}{2} e^{\nu-\lambda}; \quad \Gamma_{\phi\phi}^r = -r \sin^2 \theta e^{-\lambda} \tag{43}$$

$$\Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta; \quad \Gamma_{r\theta}^{\theta} = \Gamma_{r\phi}^{\phi} = \frac{1}{r}; \quad \Gamma_{\theta\phi}^{\phi} = \cot \theta \tag{44}$$

with curvature tensor components,

$$R_{trtr} = e^{\nu} \left(\frac{\nu''}{2} - \frac{\nu' \lambda'}{4} + \frac{\nu'^2}{4} \right) \tag{45}$$

$$R_{t\theta t\theta} = \frac{r\nu'}{2} e^{\nu-\lambda}; \quad R_{t\phi t\phi} = \frac{r\nu'}{2} \sin^2 \theta e^{\nu-\lambda} \tag{46}$$

$$R_{r\theta r\theta} = \frac{r\lambda'}{2}; \quad R_{r\phi r\phi} = \frac{r\lambda'}{2} \sin^2 \theta; \quad R_{\theta\phi\theta\phi} = r^2 \sin^2 \theta (1 - e^{-\lambda}) \tag{47}$$

Along with the following expressions:

$$(g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 = e^{-2\lambda} \left(\frac{\nu''}{2} - \frac{\nu' \lambda'}{4} + \frac{\nu'^2}{4} \right)^2 \quad (48)$$

$$(g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 = (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 = e^{-2\lambda} \frac{\nu'^2}{4r^2} \quad (49)$$

$$(g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 = (g^{rr})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 = e^{-2\lambda} \frac{\lambda'^2}{4r^2} \quad (50)$$

$$(g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 = \frac{(1 - e^{-\lambda})^2}{r^4} \quad (51)$$

For static spherically symmetric spacetime, the curvature tensor components that enter the effective field equations satisfy the following relations:

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 4 \left\{ (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right. \\ \left. + (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 + (g^{rr})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 + (g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 \right\} \quad (52)$$

$$R_{t\alpha\beta\mu} R^{t\alpha\beta\mu} = 2 \left\{ (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right\} \quad (53)$$

$$R_{r\alpha\beta\mu} R^{r\alpha\beta\mu} = 2 \left\{ (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 + (g^{rr})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 + (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 \right\} \quad (54)$$

Thus the temporal part of the effective field equations presented in [Eq. \(37\)](#) leads to

$$G_t^t + E_t^t = -\alpha \left\{ \frac{4}{3} R_{t\alpha\beta\mu} R^{t\alpha\beta\mu} - \frac{7}{12} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4R_{\mu t\nu}^t E^{\mu\nu} \right\} \\ = -\alpha \left\{ \frac{8}{3} \left[(g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right] \right. \\ - \frac{7}{3} \left[(g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right. \\ \left. + (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 + (g^{rr})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 + (g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 \right] \\ \left. - 4R_{rtr}^t E^{rr} - 4R_{\theta t\theta}^t E^{\theta\theta} - 4R_{\phi t\phi}^t E^{\phi\phi} \right\} \\ = -\alpha \left[\frac{1}{3} (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + \frac{1}{3} (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + \frac{1}{3} (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right. \\ - \frac{7}{3} (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 - \frac{7}{3} (g^{rr})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 - \frac{7}{3} (g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 \\ \left. - 4g^{tt} R_{trtr} E^{rr} - 4g^{tt} R_{t\theta t\theta} E^{\theta\theta} - 4g^{tt} R_{t\phi t\phi} E^{\phi\phi} \right] \quad (55)$$

For the background general relativity solution we have $e^\nu = e^{-\lambda} = 1 - (a/r) - (b/r^2)$, for which we readily obtain:

$$\nu' = e^\lambda \left(\frac{a}{r^2} + \frac{2b}{r^3} \right) \quad (56)$$

$$\nu'' = e^{2\lambda} \left(-\frac{2a}{r^3} + \frac{a^2 - 6b}{r^4} + \frac{4ab}{r^5} + \frac{2b^2}{r^6} \right) \quad (57)$$

$$\frac{1}{2} \times e^\nu (\nu'' + \nu'^2) = -\frac{a}{r^3} - \frac{3b}{r^4} \quad (58)$$

which immediately leads to,

$$(g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 = \left(\frac{a}{r^3} + \frac{3b}{r^4} \right)^2 \quad (59)$$

$$(g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 = (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 = \frac{1}{4} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 \quad (60)$$

$$(g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 = (g^{rr})^2 (g^{\phi\phi})^2 (R_{r\phi r\phi})^2 = \frac{1}{4} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 \quad (61)$$

$$(g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 = \left(\frac{a}{r^3} + \frac{b}{r^4} \right)^2 \quad (62)$$

Thus the right hand side of Eq. (55) can be simplified and we finally obtain,

$$\begin{aligned} & -\alpha \times \left[\frac{1}{3} (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + \frac{1}{3} (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + \frac{1}{3} (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right. \\ & \quad \left. - \frac{7}{3} (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 - \frac{7}{3} (g^{rr})^2 (g^{\phi\phi})^2 (R_{r\phi r\phi})^2 - \frac{7}{3} (g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 \right] \\ & = -\alpha \left[\frac{1}{3} \left(\frac{a}{r^3} + \frac{3b}{r^4} \right)^2 + \frac{1}{6} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 - \frac{7}{6} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 - \frac{7}{3} \left(\frac{a}{r^3} + \frac{b}{r^4} \right)^2 \right] \\ & = -\alpha \left[\frac{a^2}{3r^6} + \frac{2ab}{r^7} + \frac{3b^2}{r^8} + \frac{a^2}{6r^6} + \frac{2ab}{3r^7} + \frac{2b^2}{3r^8} - \frac{7}{6} \left(\frac{a^2}{r^6} + \frac{4ab}{r^7} + \frac{4b^2}{r^8} \right) \right. \\ & \quad \left. - \frac{7}{3} \left(\frac{a^2}{r^6} + \frac{2ab}{r^7} + \frac{b^2}{r^8} \right) \right] \\ & = -\alpha \left(-3 \frac{a^2}{r^6} - \frac{20}{3} \frac{ab}{r^7} - \frac{10}{3} \frac{b^2}{r^8} \right) \end{aligned} \quad (63)$$

Substitution of which in Eq. (55) finally leads to,

$$\begin{aligned} & G_t^t + 3\bar{\kappa}U(r) - \alpha \left(3 \frac{a^2}{r^6} + \frac{20}{3} \frac{ab}{r^7} + \frac{10}{3} \frac{b^2}{r^8} \right) - 4\alpha \left[-\bar{\kappa}e^{-\lambda}e^{-\nu} \left(\frac{a}{r^3} + \frac{3b}{r^4} \right) (U + 2P) \right. \\ & \quad \left. + \bar{\kappa} \frac{1}{2r} (U - P) e^{\lambda-\nu} \left(\frac{a}{r^2} + \frac{2b}{r^3} \right) e^{\nu-\lambda} + \bar{\kappa} \frac{1}{2r} (U - P) e^{\lambda-\nu} \left(\frac{a}{r^2} + \frac{2b}{r^3} \right) e^{\nu-\lambda} \right] = 0 \end{aligned} \quad (64)$$

Also the radial equation obtained from Eq. (37) can be simplified leading to,

$$\begin{aligned}
G_r^r &= -E_r^r - \alpha \left[\frac{1}{3} \left(\frac{a}{r^3} + \frac{3b}{r^4} \right)^2 + \frac{1}{6} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 - \frac{7}{6} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 - \frac{7}{3} \left(\frac{a}{r^3} + \frac{b}{r^4} \right)^2 \right. \\
&\quad \left. + 2e^{-\lambda} (3\bar{\kappa}U) \left(\frac{2a}{r^3} + \frac{6b}{r^4} \right) (-e^{-\nu}) + 4e^{-\lambda} \bar{\kappa} (U - P) \left(\frac{a}{r^3} + \frac{2b}{r^4} \right) (-e^\lambda) \right] \\
&= \bar{\kappa} (U + 2P) + \alpha \left(3 \frac{a^2}{r^6} + \frac{20ab}{3r^7} + \frac{10b^2}{3r^8} \right) + 4\alpha \left[3\bar{\kappa}U \left(\frac{a}{r^3} + \frac{3b}{r^4} \right) + \bar{\kappa} (U - P) \left(\frac{a}{r^3} + \frac{2b}{r^4} \right) \right]
\end{aligned} \tag{65}$$

These are the equations we have used in order to get Eq. (22) and Eq. (23) respectively.

A.3 Effective Field Equations in f(R) gravity with Gauss-Bonnet correction term: Detailed Analysis

The metric for the background $f(\mathcal{R})$ gravity spacetime can be written in the following form,

$$e^\nu = e^{-\lambda} = 1 - \frac{a}{r} - \frac{b}{r^2} + cr^2 \tag{66}$$

Then we obtain the following identities

$$e^\nu \nu' = \frac{a}{r^2} + \frac{2b}{r^3} + 2cr \tag{67}$$

$$\nu'' = e^{-2\nu} \left(-\frac{2a}{r^3} + \frac{a^2 - 6b}{r^4} + \frac{4ab}{r^5} + \frac{2b^2}{r^6} + 2c - \frac{8ac}{r} - \frac{16bc}{r^2} - 2c^2 r^2 \right) \tag{68}$$

$$\frac{1}{2} \times e^\nu (\nu'' + \nu'^2) = -\frac{a}{r^3} - \frac{3b}{r^4} + c \tag{69}$$

which immediately leads to,

$$(g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 = \left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right)^2 \tag{70}$$

$$(g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 = (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 = \frac{1}{4} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 \tag{71}$$

$$(g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 = (g^{rr})^2 (g^{\phi\phi})^2 (R_{r\phi r\phi})^2 = \frac{1}{4} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 \tag{72}$$

$$(g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 = \left(\frac{a}{r^3} + \frac{b}{r^4} - c \right)^2 \tag{73}$$

Thus we obtain the following result

$$\begin{aligned}
& -\alpha \times \left[\frac{1}{3} (g^{tt})^2 (g^{rr})^2 (R_{trtr})^2 + \frac{1}{3} (g^{tt})^2 (g^{\theta\theta})^2 (R_{t\theta t\theta})^2 + \frac{1}{3} (g^{tt})^2 (g^{\phi\phi})^2 (R_{t\phi t\phi})^2 \right. \\
& \quad \left. - \frac{7}{3} (g^{rr})^2 (g^{\theta\theta})^2 (R_{r\theta r\theta})^2 - \frac{7}{3} (g^{rr})^2 (g^{\phi\phi})^2 (R_{r\phi r\phi})^2 - \frac{7}{3} (g^{\phi\phi})^2 (g^{\theta\theta})^2 (R_{\phi\theta\phi\theta})^2 \right] \\
& = -\alpha \left[\frac{1}{3} \left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right)^2 + \frac{1}{6} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 - \frac{7}{6} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 - \frac{7}{3} \left(\frac{a}{r^3} + \frac{b}{r^4} - c \right)^2 \right] \\
& = -\alpha \left(-3 \frac{a^2}{r^6} - \frac{20}{3} \frac{ab}{r^7} - \frac{10}{3} \frac{b^2}{r^8} - \frac{20}{3} c^2 - \frac{16}{3} \frac{bc}{r^4} \right) \tag{74}
\end{aligned}$$

Using the above identity we arrive at the temporal part of the effective field equations as,

$$\begin{aligned}
G_t^t + E_t^t + [\Lambda_4 - F(\mathcal{R})] h_t^t &= -\alpha \left(-3 \frac{a^2}{r^6} - \frac{20}{3} \frac{ab}{r^7} - \frac{10}{3} \frac{b^2}{r^8} - \frac{20}{3} c^2 - \frac{16}{3} \frac{bc}{r^4} \right) \\
& + 4\alpha \left[-\bar{\kappa} e^{-\lambda} e^{-\nu} (U + 2P) \left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right) + \frac{\bar{\kappa}}{2r} (U - P) e^{\lambda-\nu} \left(\frac{a}{r^2} + \frac{2b}{r^3} + 2cr \right) e^{\nu-\lambda} \right. \\
& \quad \left. + \frac{\bar{\kappa}}{2r} (U - P) e^{\lambda-\nu} \left(\frac{a}{r^2} + \frac{2b}{r^3} + 2cr \right) e^{\nu-\lambda} \right] \tag{75}
\end{aligned}$$

Which after simplification leads to,

$$\begin{aligned}
G_t^t &= -3\bar{\kappa}U(r) + \alpha \left(3 \frac{a^2}{r^6} + \frac{20}{3} \frac{ab}{r^7} + \frac{10}{3} \frac{b^2}{r^8} + \frac{20}{3} c^2 + \frac{16}{3} \frac{bc}{r^4} \right) - [\Lambda_4 - F(\mathcal{R})] \\
& + 4\alpha \left[-\frac{3\bar{\kappa}P_0}{2r^4} \left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right) - \frac{3\bar{\kappa}P_0}{2r^4} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right) \right] \\
& = -3\bar{\kappa}U(r) + \alpha \left(3 \frac{a^2}{r^6} + \frac{20}{3} \frac{ab}{r^7} + \frac{10}{3} \frac{b^2}{r^8} + \frac{20}{3} c^2 + \frac{16}{3} \frac{bc}{r^4} \right) - [\Lambda_4 - F(\mathcal{R})] + \frac{6\alpha\bar{\kappa}P_0}{r^4} \left(\frac{2a}{r^3} + \frac{5b}{r^4} + c \right) \tag{76}
\end{aligned}$$

which leads to the following solution for $e^{-\lambda}$ as,

$$\begin{aligned}
e^{-\lambda} &= 1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + \alpha \left(-\frac{a^2}{r^4} - \frac{5}{3} \frac{ab}{r^5} - \frac{2}{3} \frac{b^2}{r^6} + 20c^2 r^2 - \frac{16}{3} \frac{bc}{r^2} \right) \\
& + [F(\mathcal{R}) - \Lambda_4] \frac{r^2}{3} + 6\alpha\bar{\kappa}P_0 \left(\frac{a}{2r^5} + \frac{b}{r^6} + \frac{c}{r^2} \right) \tag{77}
\end{aligned}$$

The radial part of effective equation turns out to be,

$$\begin{aligned}
G_r^r + E_r^r + \Lambda_4 - F(\mathcal{R}) &= -\alpha \left[\frac{1}{3} \left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right)^2 - \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 - \frac{7}{3} \left(\frac{a}{r^3} + \frac{b}{r^4} - c \right)^2 \right] \\
& - \alpha \left[-6e^{-\lambda} e^{-\nu} \bar{\kappa}U \left(\frac{2a}{r^3} + \frac{6b}{r^4} - 2c \right) - 4\bar{\kappa} (U - P) \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right) \right] \tag{78}
\end{aligned}$$

which ultimately leads to,

$$G_r^r = [F(\mathcal{R}) - \Lambda_4] + \bar{\kappa} (U + 2P) + \alpha \left[\frac{3a^2}{r^6} + \frac{10}{3} \frac{b^2}{r^8} + \frac{20}{3} c^2 + \frac{20}{3} \frac{ab}{r^7} + \frac{10}{3} \frac{bc}{r^4} \right] + 4\alpha \left[3\bar{\kappa} U(r) \left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right) + \bar{\kappa} (U - P) \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right) \right] \quad (79)$$

This again shows that $e^\nu = e^{-\lambda}$.

A.4 Low Energy Effective Action for General Relativity with Gauss-Bonnet correction term: Detailed Analysis

The temporal component of the low energy effective field equations, as presented in Eq. (10) reads,

$$\begin{aligned} G_t^t &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{2\beta\ell^2}{1+\beta} \left(R_{t\alpha\beta\mu} R^{t\alpha\beta\mu} - \frac{1}{4} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) - \frac{\beta\ell^2}{3} \left(R_{t\alpha\beta\mu} R^{t\alpha\beta\mu} - \frac{7}{16} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \\ &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{2\beta\ell^2}{1+\beta} \left\{ \left(2R_{trtr} R^{trtr} + 2R_{t\theta t\theta} R^{t\theta t\theta} + 2R_{t\phi t\phi} R^{t\phi t\phi} \right) - \frac{1}{4} \left(4R_{trtr} R^{trtr} \right. \right. \\ &\quad \left. \left. + 4R_{t\theta t\theta} R^{t\theta t\theta} + 4R_{t\phi t\phi} R^{t\phi t\phi} + 4R_{r\theta r\theta} R^{r\theta r\theta} + 4R_{r\phi r\phi} R^{r\phi r\phi} + 4R_{\theta\phi\theta\phi} R^{\theta\phi\theta\phi} \right) \right\} \\ &\quad - \frac{\beta\ell^2}{3} \left\{ \left(2R_{trtr} R^{trtr} + 2R_{t\theta t\theta} R^{t\theta t\theta} + 2R_{t\phi t\phi} R^{t\phi t\phi} \right) - \frac{7}{16} \left(4R_{trtr} R^{trtr} \right. \right. \\ &\quad \left. \left. + 4R_{t\theta t\theta} R^{t\theta t\theta} + 4R_{t\phi t\phi} R^{t\phi t\phi} + 4R_{r\theta r\theta} R^{r\theta r\theta} + 4R_{r\phi r\phi} R^{r\phi r\phi} + 4R_{\theta\phi\theta\phi} R^{\theta\phi\theta\phi} \right) \right\} \\ &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left(R_{trtr} R^{trtr} + R_{t\theta t\theta} R^{t\theta t\theta} + R_{t\phi t\phi} R^{t\phi t\phi} \right) \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left(R_{r\theta r\theta} R^{r\theta r\theta} + R_{r\phi r\phi} R^{r\phi r\phi} + R_{\theta\phi\theta\phi} R^{\theta\phi\theta\phi} \right) \end{aligned} \quad (80)$$

Again starting from the expressions for curvature components with metric elements $e^\nu = e^{-\lambda} = 1 - (a/r) - (b/r^2)$, we readily obtain,

$$\begin{aligned} G_t^t &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left[\left(\frac{a}{r^3} + \frac{3b}{r^4} \right)^2 + \frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 \right] \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left[\frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 + \left(\frac{a}{r^3} + \frac{b}{r^4} \right)^2 \right] \end{aligned} \quad (81)$$

Substituting for the expression for G_t^t , we get,

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} (r e^{-\lambda}) - \frac{1}{r^2} &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left(\frac{3}{2} \frac{a^2}{r^6} + \frac{8ab}{r^7} + \frac{11b^2}{r^8} \right) \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left(\frac{3a^2}{2r^6} + \frac{4ab}{r^7} + \frac{3b^2}{r^8} \right) \end{aligned} \quad (82)$$

which can be integrated to yield,

$$e^{-\lambda} = 1 - \frac{2GM + Q_0}{r} + \frac{2}{\ell} \frac{1-\beta}{1+\beta} \frac{1}{r} \int {}^{(2)}\chi_t^t(r) r^2 dr + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left(-\frac{a^2}{2r^4} - \frac{2ab}{r^5} - \frac{11b^2}{5r^6} \right) + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left(-\frac{a^2}{2r^4} - \frac{ab}{r^5} - \frac{3b^2}{5r^6} \right) \quad (83)$$

The radial equation leads to,

$$\begin{aligned} G_r^r &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_r^r + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} (R_{r\theta r\theta} R^{r\theta r\theta} + R_{rt rt} R^{rt rt} + R_{r\phi r\phi} R^{r\phi r\phi}) \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} (R_{t\theta t\theta} R^{t\theta t\theta} + R_{t\phi t\phi} R^{t\phi t\phi} + R_{\theta\phi\theta\phi} R^{\theta\phi\theta\phi}) \\ &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_r^r + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left[\frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 + \left(\frac{a}{r^3} + \frac{3b}{r^4} \right)^2 \right] \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left[\frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 + \left(\frac{a}{r^3} + \frac{b}{r^4} \right)^2 \right] \\ &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_r^r + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left(\frac{3}{2} \frac{a^2}{r^6} + \frac{8ab}{r^7} + \frac{11b^2}{r^8} \right) \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left(\frac{3a^2}{2r^6} + \frac{4ab}{r^7} + \frac{3b^2}{r^8} \right) \end{aligned} \quad (84)$$

hence $e^\nu = e^{-\lambda}$. However from the above structure the corrections are not immediately obvious. For that we note using ${}^{(2)}E_{\mu\nu} = -(2/\ell) {}^{(2)}\bar{\chi}_{\mu\nu}$ the field equations transform to,

$$\begin{aligned} G_\sigma^\rho &= \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\bar{\chi}_\sigma^\rho - \frac{\beta\ell^2}{3} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{7}{16} \delta_\sigma^\rho R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \\ &= -E_\sigma^\rho + \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_\sigma^\rho + \frac{4\beta^2\ell^2}{6(1-\beta^2)} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{1}{4} \delta_\sigma^\rho R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \\ &\quad - \frac{\beta\ell^2}{3} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{7}{16} \delta_\sigma^\rho R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \end{aligned} \quad (85)$$

Thus keeping terms upto linear order in β the temporal component turns out to be,

$$\begin{aligned} G_t^t + E_t^t &= \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t - \frac{\beta\ell^2}{12} (R_{trtr} R^{trtr} + R_{t\theta t\theta} R^{t\theta t\theta} + R_{t\phi t\phi} R^{t\phi t\phi}) \\ &\quad + \frac{7\beta\ell^2}{12} (R_{r\theta r\theta} R^{r\theta r\theta} + R_{r\phi r\phi} R^{r\phi r\phi} + R_{\theta\phi\theta\phi} R^{\theta\phi\theta\phi}) \\ &= \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t - \frac{\beta\ell^2}{12} \left[\left(\frac{a}{r^3} + \frac{3b}{r^4} \right)^2 + \frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 \right] + \frac{7\beta\ell^2}{12} \left[\frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} \right)^2 + \left(\frac{a}{r^3} + \frac{b}{r^4} \right)^2 \right] \\ &= \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t - \frac{\beta\ell^2}{12} \left(\frac{3}{2} \frac{a^2}{r^6} + \frac{8ab}{r^7} + \frac{11b^2}{r^8} \right) + \frac{7\beta\ell^2}{12} \left(\frac{3a^2}{2r^6} + \frac{4ab}{r^7} + \frac{3b^2}{r^8} \right) \end{aligned} \quad (86)$$

From which the metric element $e^{-\lambda}$ would turn out to be,

$$e^{-\lambda} = 1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + \frac{4\beta}{\ell(1+\beta)} \frac{1}{r} \int {}^{(2)}\chi_t^t r^2 dr - \frac{\beta\ell^2}{12} \left(3\frac{a^2}{r^4} + \frac{5ab}{r^5} + \frac{2b^2}{r^6} \right) \quad (87)$$

We will turn to the last bit of derivation, which involves the low energy equation with $f(\mathcal{R})$ gravity in the bulk.

A.5 Low Energy Effective Action for $f(\mathcal{R})$ gravity with Gauss-Bonnet correction term: Detailed Analysis

The low energy effective equation with $f(\mathcal{R})$ gravity corresponds to,

$$\begin{aligned} G_\sigma^\rho &= [F(\mathcal{R}) - \Lambda_4] \delta_\sigma^\rho + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_\sigma^\rho + \frac{2\beta\ell^2}{1+\beta} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{1}{4} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \\ &\quad - \frac{\beta\ell^2}{3} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{7}{16} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \end{aligned} \quad (88)$$

Now considering the temporal component for background metric element $e^\nu = e^{-\lambda} = 1 - (a/r) - (b/r^2) + cr^2$ we arrive at,

$$\begin{aligned} G_t^t &= [F(\mathcal{R}) - \Lambda_4] + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} (R_{trtr} R^{trtr} + R_{t\theta t\theta} R^{t\theta t\theta} + R_{t\phi t\phi} R^{t\phi t\phi}) \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} (R_{r\theta r\theta} R^{r\theta r\theta} + R_{r\phi r\phi} R^{r\phi r\phi} + R_{\theta\phi\theta\phi} R^{\theta\phi\theta\phi}) \\ &= [F(\mathcal{R}) - \Lambda_4] + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left[\left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right)^2 + \frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 \right] \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left[\frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 + \left(\frac{a}{r^3} + \frac{b}{r^4} - c \right)^2 \right] \end{aligned} \quad (89)$$

writing explicitly we obtain,

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} (re^{-\lambda}) - \frac{1}{r^2} &= [F(\mathcal{R}) - \Lambda_4] + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\chi_t^t + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left[\frac{3a^2}{2r^6} + \frac{11b^2}{r^8} + \frac{8ab}{r^7} + 3c^2 - \frac{2bc}{r^4} \right] \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left[\frac{3a^2}{2r^6} + \frac{3b^2}{r^8} + \frac{4ab}{r^7} + 3c^2 - \frac{2bc}{r^4} \right] \end{aligned} \quad (90)$$

The metric element can be solved as,

$$\begin{aligned} e^{-\lambda} &= 1 - \frac{2GM + Q_0}{r} + \frac{F(\mathcal{R}) - \Lambda_4}{3} r^2 + \frac{2}{\ell} \frac{1-\beta}{1+\beta} \frac{1}{r} \int {}^{(2)}\chi_t^t r^2 dr \\ &\quad + \frac{(23-\beta)\beta\ell^2}{12(1+\beta)} \left[-\frac{a^2}{2r^4} - \frac{11b^2}{5r^6} - \frac{2ab}{r^5} + c^2 r^2 - \frac{2bc}{r^2} \right] \\ &\quad + \frac{(7\beta-17)\beta\ell^2}{12(1+\beta)} \left[-\frac{a^2}{2r^4} - \frac{3b^2}{5r^6} - \frac{ab}{r^5} + c^2 r^2 - \frac{2bc}{r^2} \right] \end{aligned} \quad (91)$$

In this case as well we note, $^{(2)}E_\nu^\mu = -(2/\ell) {}^{(2)}\bar{\chi}_\nu^\mu$. Then,

$$\begin{aligned}
G_\sigma^\rho &= [F(\mathcal{R}) - \Lambda_4] \delta_\sigma^\rho + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\bar{\chi}_\sigma^\rho - \frac{\beta\ell^2}{3} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{7}{16} \delta_\sigma^\rho R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \\
&= -E_\sigma^\rho + [F(\mathcal{R}) - \Lambda_4] \delta_\sigma^\rho + \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_\sigma^\rho + \frac{4\beta^2\ell^2}{6(1-\beta^2)} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{1}{4} \delta_\sigma^\rho R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \\
&\quad - \frac{\beta\ell^2}{3} \left(R_{\sigma\alpha\beta\mu} R^{\rho\alpha\beta\mu} - \frac{7}{16} \delta_\sigma^\rho R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right)
\end{aligned} \tag{92}$$

Thus keeping terms upto linear order in β the temporal component turns out to be,

$$\begin{aligned}
G_t^t + E_t^t &= [F(\mathcal{R}) - \Lambda_4] + \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t - \frac{\beta\ell^2}{12} (R_{trtr} R^{trtr} + R_{t\theta t\theta} R^{t\theta t\theta} + R_{t\phi t\phi} R^{t\phi t\phi}) \\
&\quad + \frac{7\beta\ell^2}{12} (R_{r\theta r\theta} R^{r\theta r\theta} + R_{r\phi r\phi} R^{r\phi r\phi} + R_{\theta\phi\theta\phi} R^{\theta\phi\theta\phi}) \\
&= [F(\mathcal{R}) - \Lambda_4] + \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t - \frac{\beta\ell^2}{12} \left[\left(\frac{a}{r^3} + \frac{3b}{r^4} - c \right)^2 + \frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 \right] \\
&\quad + \frac{7\beta\ell^2}{12} \left[\frac{1}{2} \left(\frac{a}{r^3} + \frac{2b}{r^4} + 2c \right)^2 + \left(\frac{a}{r^3} + \frac{b}{r^4} - c \right)^2 \right] \\
&= [F(\mathcal{R}) - \Lambda_4] + \frac{4\beta}{\ell(1+\beta)} {}^{(2)}\chi_t^t - \frac{\beta\ell^2}{12} \left[\frac{3a^2}{2r^6} + \frac{11b^2}{r^8} + \frac{8ab}{r^7} + 3c^2 - \frac{2bc}{r^4} \right] \\
&\quad + \frac{7\beta\ell^2}{12} \left[\frac{3a^2}{2r^6} + \frac{3b^2}{r^8} + \frac{4ab}{r^7} + 3c^2 - \frac{2bc}{r^4} \right]
\end{aligned} \tag{93}$$

For which the metric element turns out to be,

$$\begin{aligned}
e^{-\lambda} &= 1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + \frac{F(\mathcal{R}) - \Lambda_4}{3} r^2 + \frac{4\beta}{\ell(1+\beta)} \frac{1}{r} \int {}^{(2)}\chi_t^t r^2 dr \\
&\quad - \frac{\beta\ell^2}{12} \left[\frac{3a^2}{r^4} + \frac{2b^2}{r^6} + \frac{5ab}{r^5} - 6c^2 r^2 - \frac{12bc}{r^2} \right] \\
&= 1 - \frac{2GM + Q_0}{r} - \frac{3\bar{\kappa}P_0}{2r^2} + \frac{F(\mathcal{R}) - \Lambda_4}{3} r^2 + \frac{16\alpha}{\ell^3} \frac{1}{r} \int {}^{(2)}\chi_t^t r^2 dr \\
&\quad - \frac{\alpha}{3} \left[\frac{3a^2}{r^4} + \frac{2b^2}{r^6} + \frac{5ab}{r^5} - 6c^2 r^2 - \frac{12bc}{r^2} \right]
\end{aligned} \tag{94}$$

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